SUBEXPONENTIAL SIZE HITTING SETS FOR BOUNDED DEPTH MULTILINEAR FORMULAS

Ben Lee Volk (Tel Aviv University)

Joint work with

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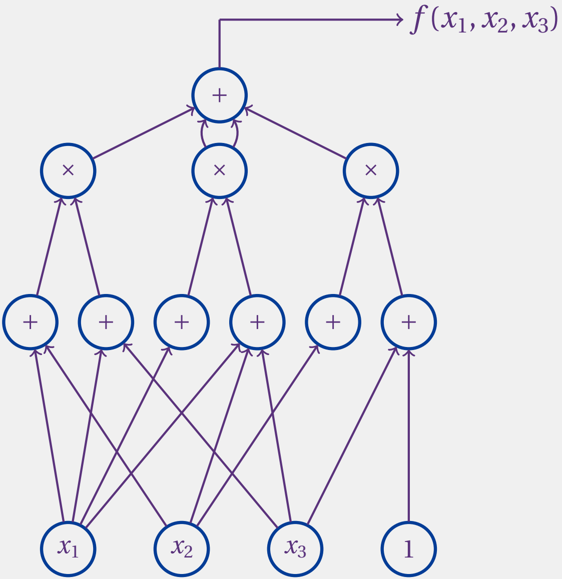
Amir Shpilka (Tel Aviv University)

ARITHMETIC CIRCUITS



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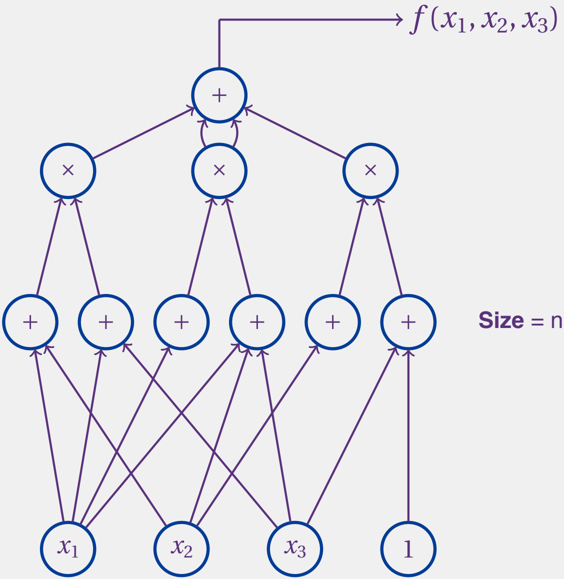

Size = number of gates

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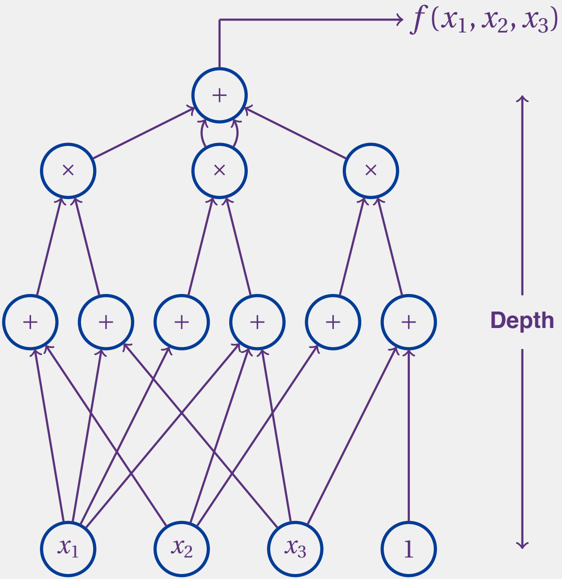
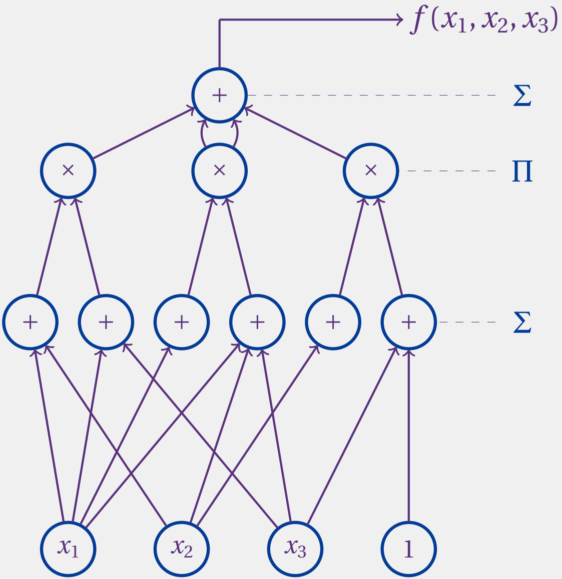
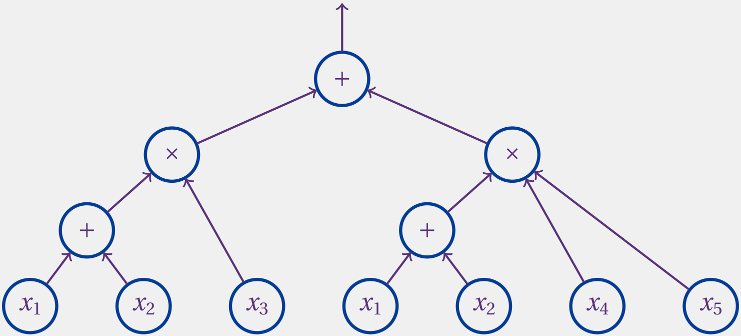


Arithmetic Circuits



Arithmetic Formulas

Underlying graph is a tree.



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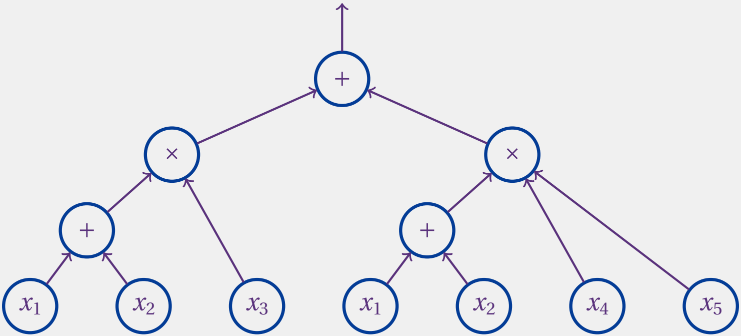

Multilinear Formula: every node computes a multilinear polynomial.

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Given $C(x_1, \ldots, x_n)$,
decide *deterministically* whether $C \equiv 0$.

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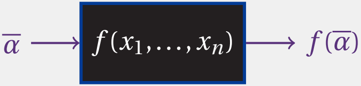

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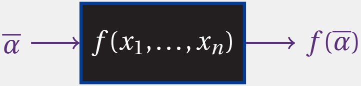

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$$\overline{\alpha} \longrightarrow \boxed{f(x_1, \ldots, x_n)} \longrightarrow f(\overline{\alpha})$$

Black Box PIT $\equiv$ explicit hitting set.
Hitting Set for class $\mathcal{C}$: A set $\mathcal{H} \subseteq \mathbb{F}^n$ such that for every non-zero $f \in \mathcal{C}$ there exists $\overline{\alpha} \in \mathcal{H}$ such that $f(\overline{\alpha}) \neq 0$.

RELATED WORK

PIT for bounded depth circuits:

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Model	Running time

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Depth 3 circuits Bounded top fan-in	Poly time, BB	**[Kayal-Saraf], [Saxena-Seshadri]**

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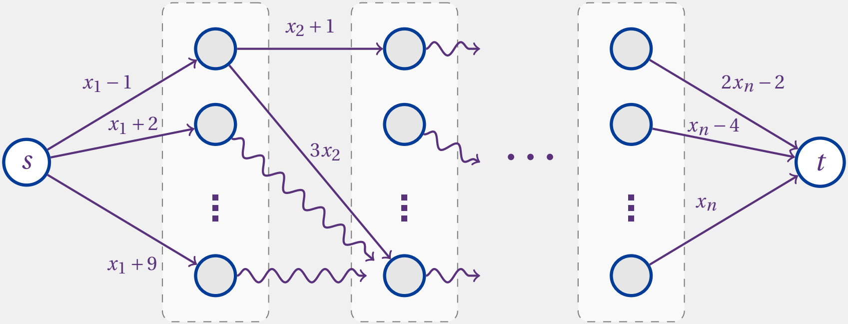
(slightly better lower bounds known but not via hitting sets)

PROOF TECHNIQUE

"Reduction" to *Read Once Algebraic Branching Program*:



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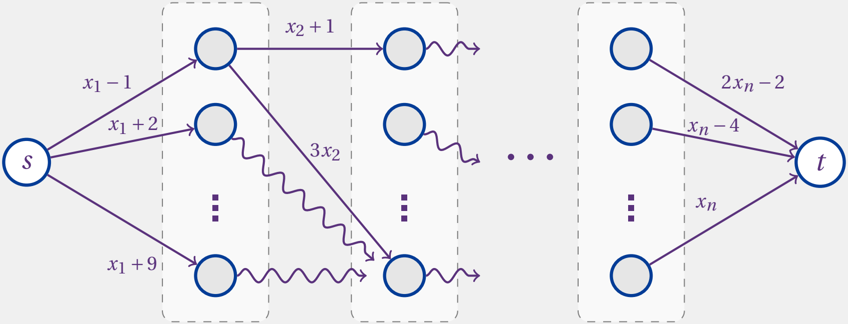

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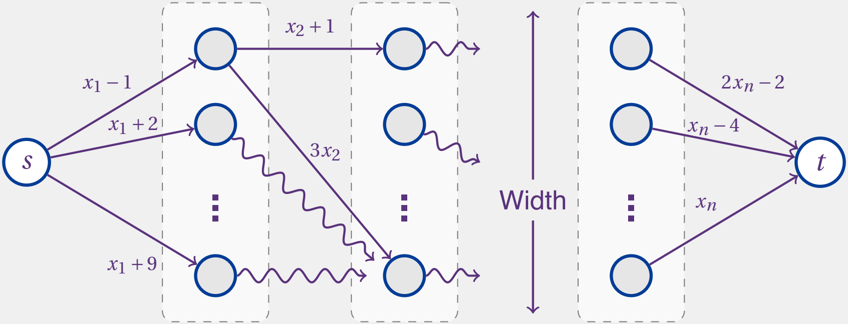

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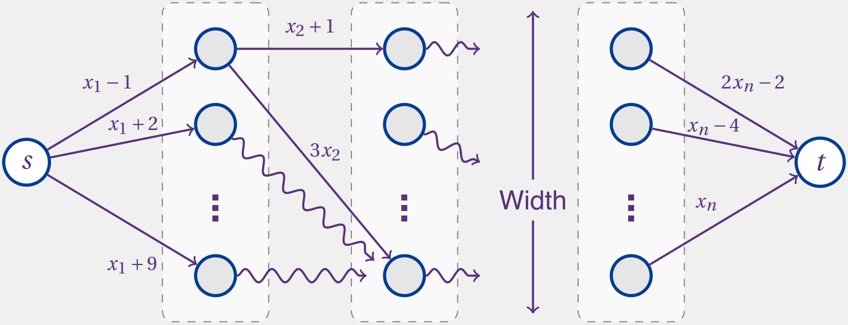
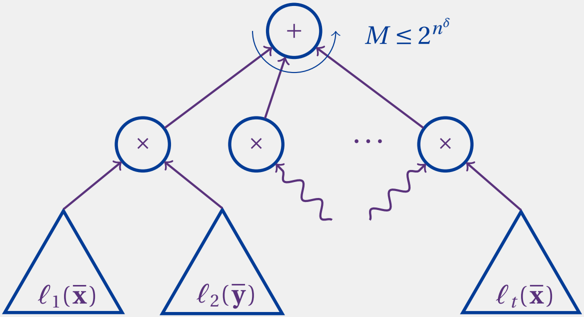


Theorem [FS13, FSS14, AGKS14]: $\exists$ explicit hitting set for ROABPs of width w of size $\text{poly}(n, w)^{O(\log n)}$.

DEPTH 3 MULTILINEAR FORMULAS: $\Sigma\Pi\Sigma$

Sum of products of linear functions with disjoint support:



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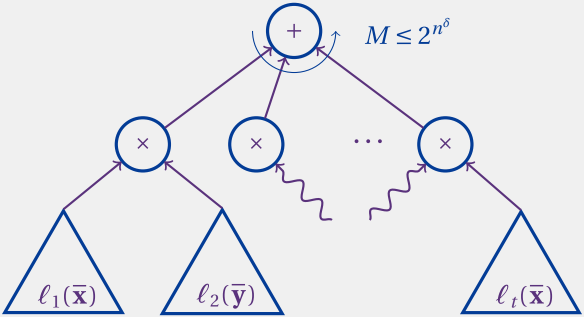

... if every linear function has only 1 variable in its support, the polynomial is computed by width M ROABP.

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Now do the same for $S_2, S_3, \ldots$

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Problems:

- What about linear functions which contain a lot of variables?
- How to find the partition?

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Solution to Problem #1: get rid of linear functions with large support (more than n^{ε} is a problem)

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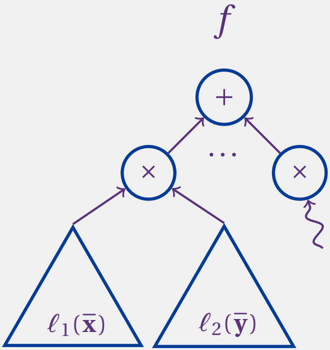

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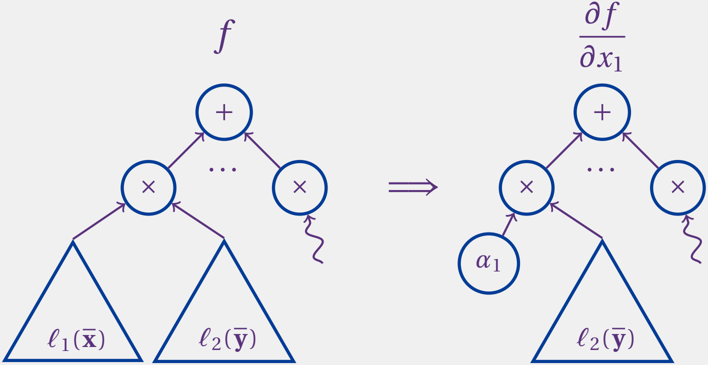

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Applied repeatedly, each query to the derivative is simulated by 2^t queries to f.

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Deterministic version: Partition vars according to n^{δ}-wise independent family of hash functions.

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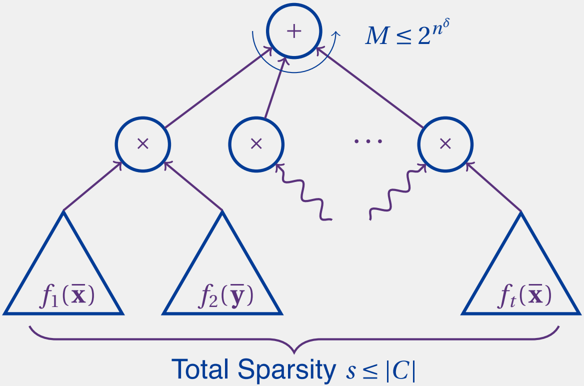
Optimizing parameters: $|\mathcal{H}| = 2^{\tilde{O}(n^{2/3+2\delta/3})}$.
Lower bound: set $\delta = 1/2 - O(\log\log n/\log n)$. Find non-zero polynomial which vanishes over $\mathcal{H}$.

DEPTH 4 MULTILINEAR FORMULAS: $\Sigma\Pi\Sigma\Pi$

Sum of products of sparse polynomials with disjoint support:



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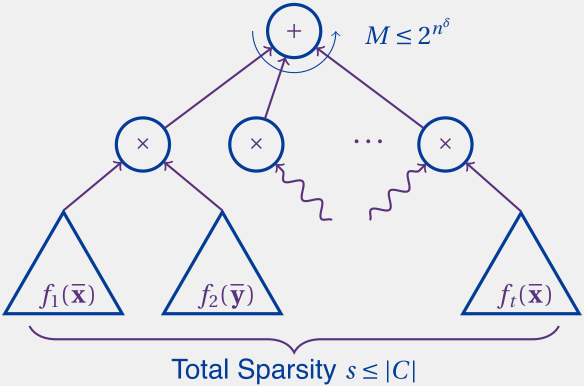

... what has changed?

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- But: for every multilinear polynomial f and variable x, either setting $x = 0$ or taking derivative with respect to x reduces the sparsity by at least half

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- But: for every multilinear polynomial f and variable x, either setting $x = 0$ or taking derivative with respect to x reduces the sparsity by at least half
- Similar argument as before: we only care about *high-support* polynomials. $\exists$ var x_i such that either in $f|_{x_i=0}$ or $\frac{\partial f}{\partial x_i}$ the total sparsity of bad polynomials is reduced by a factor of $1 - \frac{1}{2n^{1-\varepsilon}}$

REDUCING BOTTOM SUPPORT: DEPTH 4 VERSION

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- But: for every multilinear polynomial f and variable x, either setting $x = 0$ or taking derivative with respect to x reduces the sparsity by at least half
- Similar argument as before: we only care about *high-support* polynomials. $\exists$ var x_i such that either in $f|_{x_i=0}$ or $\frac{\partial f}{\partial x_i}$ the total sparsity of bad polynomials is reduced by a factor of $1 - \frac{1}{2n^{1-\varepsilon}}$
- Repeat $O(n^{1-\varepsilon} \log s)$ times to eliminate all high-support polynomials

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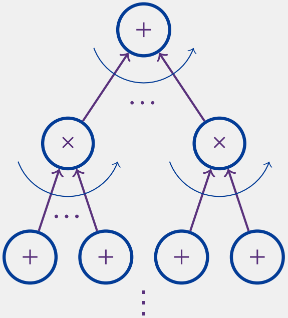
- Continue as before

REGULAR BOUNDED-DEPTH FORMULAS

[KSS14]

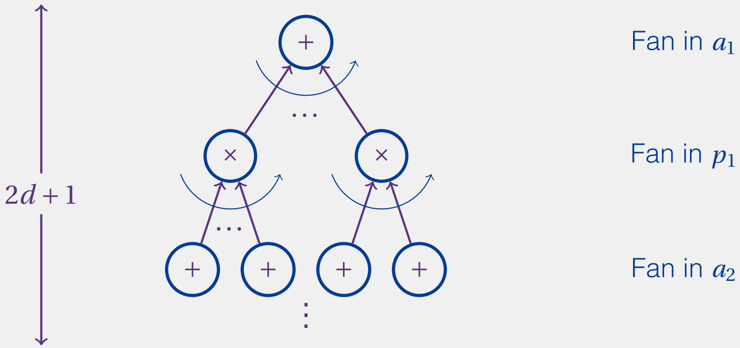


Fan in a_1

Fan in p_1

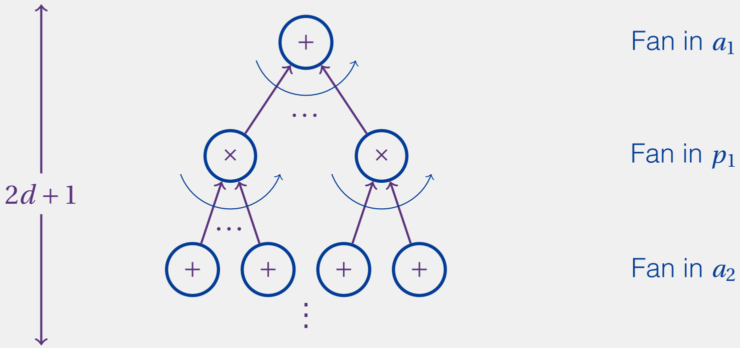
Fan in a_2

Regular Bounded-Depth Formulas



- $d+1$ levels labeled '+', d labeled '×'

REGULAR BOUNDED-DEPTH FORMULAS



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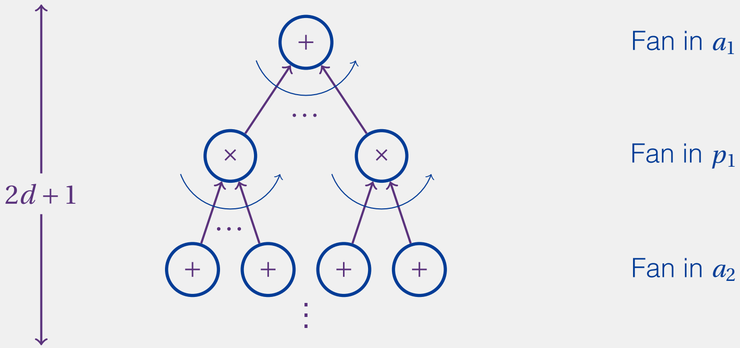

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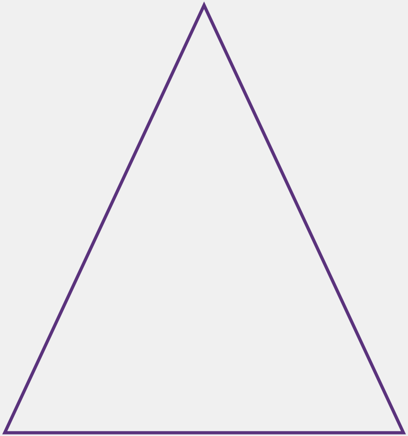
(actually fan-in of + gates is not that important)

REDUCTION TO DEPTH 4



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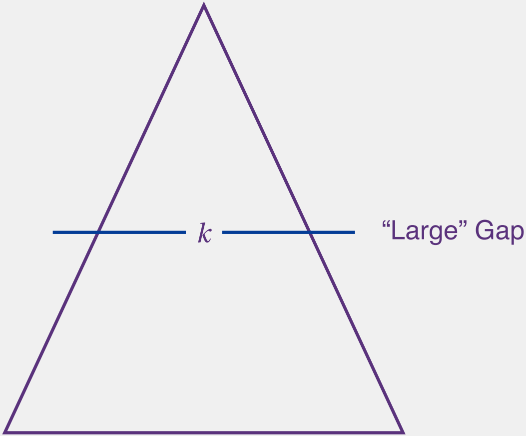

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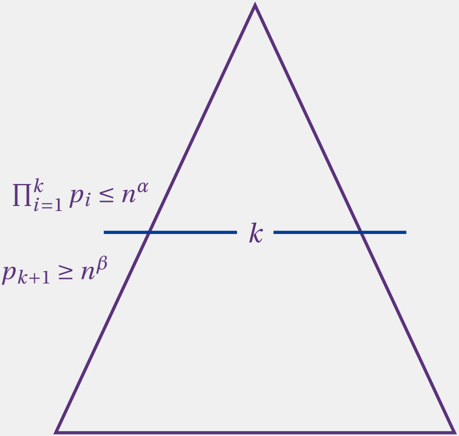

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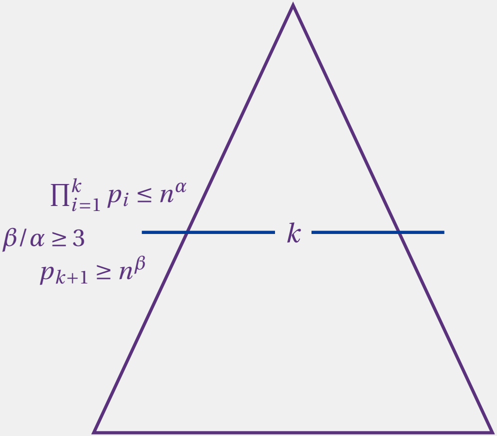
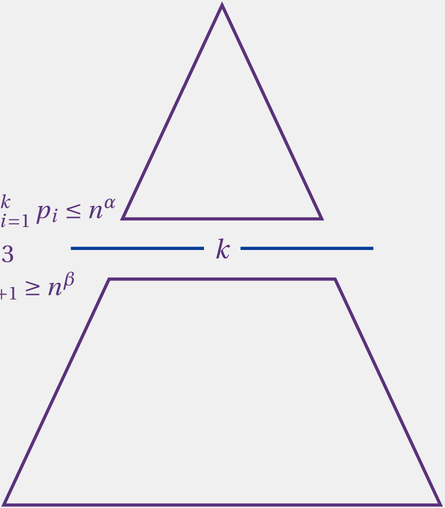


Reduction to Depth 4

$\prod_{i=1}^{k} p_i \leq n^{\alpha}$

$\beta / \alpha \geq 3$

$p_{k+1} \geq n^{\beta}$



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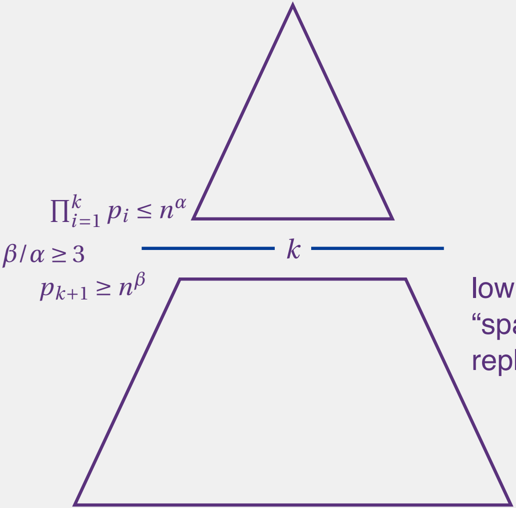

lower part: $\deg \leq n^{1-1/\exp(d)}$
"sparse" polynomial
replace w/ subexp. $\Sigma\Pi$ ckt

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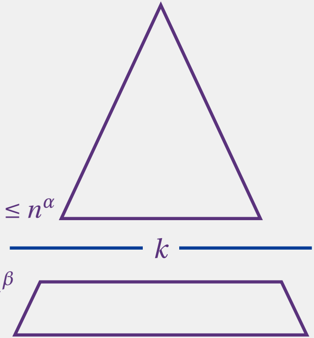

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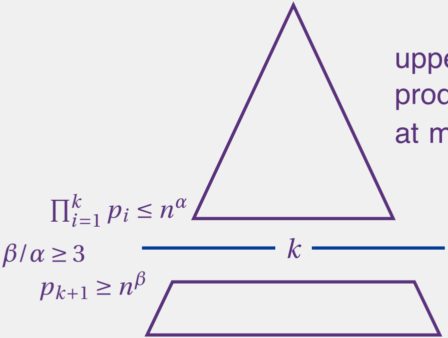

upper part: expand all products from level k upwards, at most $|C|^{\prod_{i=1}^{k} p_i}$ summands

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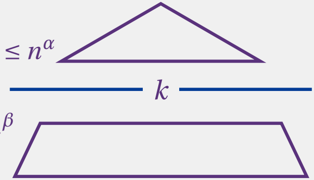

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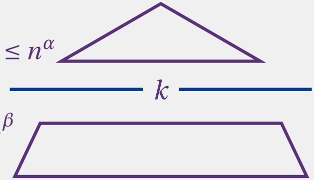

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such a large gap is required to match the depth 4 parameters. will be nice to improve.

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THANK YOU